

# SEQUENTIALLY COHEN-MACAULAY REES ALGEBRAS

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**ABSTRACT.** This paper studies the question of when the Rees algebras associated to arbitrary filtration of ideals are sequentially Cohen-Macaulay. Although this problem has been already investigated by [CGT], their situation is quite a bit of restricted, so we are eager to try the generalization of their results.

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## 1. INTRODUCTION

The notion of sequentially Cohen-Macaulay property was originally introduced by R. P. Stanley ([St]) for Stanley-Reisner algebras and then it has been furiously explored by many researchers, say D. T. Cuong, N. T. Cuong, S. Goto, P. Schenzel and others (see [CC, CGT, GHS, Sch]), from the view point of not only combinatorics, but also commutative algebra. The purpose of this paper is to investigate the question of when the Rees algebras are sequentially Cohen-Macaulay, which has a previous research by [CGT]. In [CGT] they gave a characterization of the sequentially Cohen-Macaulay Rees algebras of  $\mathfrak{m}$ -primary ideals ([CGT, Theorem 5.2, Theorem 5.3]). However their situation is not entirely satisfactory, so we are eager to analyze the case where the ideal is not necessarily  $\mathfrak{m}$ -primary. More generally we want to deal with the sequentially Cohen-Macaulayness of the Rees modules since the sequentially Cohen-Macaulay property is defined for any finite modules over a Noetherian ring. Thus the main problem of this paper is when the Rees modules associated to arbitrary filtration of modules are sequentially Cohen-Macaulay.

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Let  $R$  be a commutative Noetherian ring,  $M \neq (0)$  a finitely generated  $R$ -module with  $d = \dim_R M < \infty$ . Then we consider a filtration

$$\mathcal{D} : D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \dots \subsetneq D_\ell = M$$

of  $R$ -submodules of  $M$ , which we call *the dimension filtration of  $M$* , if  $D_{i-1}$  is the largest  $R$ -submodule of  $D_i$  with  $\dim_R D_{i-1} < \dim_R D_i$  for  $1 \leq i \leq \ell$ , here  $\dim_R(0) = -\infty$  for convention. We note here that our notion of dimension filtration is based on [GHS] and slightly different from that of the original one given by P. Schenzel ([Sch]), however let us adopt the above definition throughout this paper. Then we say that  $M$  is a *sequentially Cohen-Macaulay  $R$ -module*, if the quotient module  $C_i = D_i/D_{i-1}$  of  $D_i$  is a Cohen-Macaulay  $R$ -module for every  $1 \leq i \leq \ell$ . In particular, a Noetherian ring  $R$  is called a *sequentially Cohen-Macaulay ring*, if  $\dim R < \infty$  and  $R$  is a sequentially Cohen-Macaulay module over itself.

Let us now state our results, explaining how this paper is organized. In Section 2 we sum up the notions of the sequentially Cohen-Macaulay properties and filtrations of ideals and modules. In Section 3 we shall give the proofs of the main results of this paper, which are stated as follows.

Suppose that  $R$  is a local ring with maximal ideal  $\mathfrak{m}$ . Let  $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$  be a filtration of ideals of  $R$  such that  $F_1 \neq R$ ,  $\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}$  a  $\mathcal{F}$ -filtration of  $R$ -submodules of  $M$ . Then we put

$$\mathcal{R} = \sum_{n \geq 0} F_n t^n \subseteq R[t], \quad \mathcal{R}' = \sum_{n \in \mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}], \quad \mathcal{G} = \mathcal{R}'/t^{-1}\mathcal{R}'$$

and call them *the Rees algebra, the extended Rees algebra and the associated graded ring of  $\mathcal{F}$* , respectively. Similarly we set

$$\mathcal{R}(\mathcal{M}) = \sum_{n \geq 0} t^n \otimes M_n \subseteq R[t] \otimes_R M, \quad \mathcal{R}'(\mathcal{M}) = \sum_{n \in \mathbb{Z}} t^n \otimes M_n \subseteq R[t, t^{-1}] \otimes_R M$$

and

$$\mathcal{G}(\mathcal{M}) = \mathcal{R}'(\mathcal{M})/t^{-1}\mathcal{R}'(\mathcal{M})$$

which we call *the Rees module, the extended Rees module and the associated graded module of  $\mathcal{M}$* , respectively. Here  $t$  stands for an indeterminate over  $R$ . We also assume that  $\mathcal{R}$  is a Noetherian ring and  $\mathcal{R}(\mathcal{M})$  is a finitely generated  $\mathcal{R}$ -module. Set

$$\mathcal{D}_i = \{M_n \cap D_i\}_{n \in \mathbb{Z}}, \quad \mathcal{C}_i = \{[(M_n \cap D_i) + D_{i-1}]/D_{i-1}\}_{n \in \mathbb{Z}}.$$

for every  $1 \leq i \leq \ell$ . Then  $\mathcal{D}_i$  (resp.  $\mathcal{C}_i$ ) is a  $\mathcal{F}$ -filtration of  $R$ -submodules of  $D_i$  (resp.  $C_i$ ). With this notation the main results of this paper are the following, which are the natural generalization of the results [CGT, Theorem 5.2, Theorem 5.3].

**Theorem 1.1.** *The following conditions are equivalent.*

- (1)  $\mathcal{R}'(\mathcal{M})$  is a sequentially Cohen-Macaulay  $\mathcal{R}'$ -module.
- (2)  $\mathcal{G}(\mathcal{M})$  is a sequentially Cohen-Macaulay  $\mathcal{G}$ -module and  $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $\mathcal{G}(\mathcal{M})$ .

When this is the case,  $M$  is a sequentially Cohen-Macaulay  $R$ -module.

Let  $\mathfrak{M}$  be a unique graded maximal ideal of  $\mathcal{R}$ . We set

$$a(N) = \max\{n \in \mathbb{Z} \mid [H_{\mathfrak{M}}^t(N)]_n \neq (0)\}$$

for a finitely generated graded  $\mathcal{R}$ -module  $N$  of dimension  $t$ , and call it *the  $a$ -invariant of  $N$*  (see [GW, DEFINITION (3.1.4)]). Here  $\{[H_{\mathfrak{M}}^t(N)]_n\}_{n \in \mathbb{Z}}$  stands for the homogeneous

components of the  $t$ -th graded local cohomology module  $H_{\mathfrak{M}}^t(N)$  of  $N$  with respect to  $\mathfrak{M}$ .

**Theorem 1.2.** *Suppose that  $M$  is a sequentially Cohen-Macaulay  $R$ -module and  $F_1 \not\subseteq \mathfrak{p}$  for every  $\mathfrak{p} \in \text{Ass}_R M$ . Then the following conditions are equivalent.*

- (1)  $\mathcal{R}(\mathcal{M})$  is a sequentially Cohen-Macaulay  $\mathcal{R}$ -module.
- (2)  $\mathcal{G}(\mathcal{M})$  is a sequentially Cohen-Macaulay  $\mathcal{G}$ -module,  $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $\mathcal{G}(\mathcal{M})$  and  $\mathfrak{a}(\mathcal{G}(\mathcal{C}_i)) < 0$  for every  $1 \leq i \leq \ell$ .

When this is the case,  $\mathcal{R}'(\mathcal{M})$  is a sequentially Cohen-Macaulay  $\mathcal{R}'$ -module.

In Section 4 we focus our attention on the case of graded rings. In the last section we will explore the application of Theorem 4.5 to the Stanley-Reisner algebras of shellable complexes (Theorem 5.2).

## 2. PRELIMINARIES

In this section we summarize some basic results on sequentially Cohen-Macaulay properties and filtration of ideals and modules, which we will use throughout this paper.

Let  $R$  be a Noetherian ring,  $M \neq (0)$  a finitely generated  $R$ -module of dimension  $d$ . We put

$$\text{Assh}_R M = \{\mathfrak{p} \in \text{Supp}_R M \mid \dim R/\mathfrak{p} = d\}.$$

For each  $n \in \mathbb{Z}$ , there exists the largest  $R$ -submodule  $M_n$  of  $M$  with  $\dim_R M_n \leq n$ . Let

$$\begin{aligned} \mathcal{S}(M) &= \{\dim_R N \mid N \text{ is an } R\text{-submodule of } M, N \neq (0)\} \\ &= \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \text{Ass}_R M\}. \end{aligned}$$

We set  $\ell = \#\mathcal{S}(M)$  and write  $\mathcal{S}(M) = \{d_1 < d_2 < \dots < d_\ell = d\}$ . Let  $D_i = M_{d_i}$  for each  $1 \leq i \leq \ell$ . We then have a filtration

$$\mathcal{D} : D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \dots \subsetneq D_\ell = M$$

of  $R$ -submodules of  $M$ , which we call *the dimension filtration of  $M$* . We put  $C_i = D_i/D_{i-1}$  for every  $1 \leq i \leq \ell$ .

**Definition 2.1** ([Sch, St]). We say that  $M$  is a *sequentially Cohen-Macaulay  $R$ -module*, if  $C_i$  is Cohen-Macaulay for every  $1 \leq i \leq \ell$ . The ring  $R$  is called a *sequentially Cohen-Macaulay ring*, if  $\dim R < \infty$  and  $R$  is a sequentially Cohen-Macaulay module over itself.

The typical examples of sequentially Cohen-Macaulay ring is the Stanley-Reisner algebra  $k[\Delta]$  of a shellable complex  $\Delta$  over a field  $k$ . Also every one-dimensional Noetherian local ring is sequentially Cohen-Macaulay. Moreover, if  $M$  is a Cohen-Macaulay module over a Noetherian local ring, then  $M$  is sequentially Cohen-Macaulay, and the converse holds if  $M$  is unmixed.

Firstly let us note the non-zerodivisor characterization of sequentially Cohen-Macaulay modules.

**Proposition 2.2.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring,  $M \neq (0)$  a finitely generated  $R$ -module. Let  $x \in \mathfrak{m}$  be a non-zerodivisor on  $M$ . Then the following conditions are equivalent.*

- (1)  $M$  is a sequentially Cohen-Macaulay  $R$ -module.
- (2)  $M/xM$  is a sequentially Cohen-Macaulay  $R/(x)$ -module and  $\{D_i/xD_i\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $M/xM$ .

*Proof.* Notice that  $x \in \mathfrak{m}$  is a non-zerodivisor on  $C_i$  and  $D_i$  for all  $1 \leq i \leq \ell$  (See [Sch, Corollary 2.3]). Therefore we get a filtration

$$D_0/xD_0 = (0) \subsetneq D_1/xD_1 \subsetneq \cdots \subsetneq D_\ell/xD_\ell = M/xM$$

of  $R/(x)$ -submodules of  $M/xM$ . Then the assertion is a direct consequence of [GHS, Theorem 2.3].  $\square$

The implication (2)  $\Rightarrow$  (1) is not true without the condition that  $\{D_i/xD_i\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $M/xM$ . For instance, let  $R$  be a 2-dimensional Noetherian local domain of depth 1. Then  $R/(x)$  is sequentially Cohen-Macaulay for every  $0 \neq x \in R$ , but  $R$  is not. Besides this, let  $I$  be an  $\mathfrak{m}$ -primary ideal in a regular local ring  $(R, \mathfrak{m})$  of dimension 2. Then  $I$  is not a sequentially Cohen-Macaulay  $R$ -module, even though  $I/xI$  is, where  $0 \neq x \in \mathfrak{m}$ . These examples show that [Sch, Theorem 4.7] is not true in general.

From now on, we shall quickly review some preliminaries on filtrations of ideals and modules. Let  $R$  be a commutative ring,  $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$  a filtration of ideals of  $R$ , that is,  $F_n$  is an ideal of  $R$ ,  $F_n \supseteq F_{n+1}$ ,  $F_m F_n \subseteq F_{m+n}$  for all  $m, n \in \mathbb{Z}$  and  $F_0 = R$ . Then we put

$$\mathcal{R} = \mathcal{R}(\mathcal{F}) = \sum_{n \geq 0} F_n t^n \subseteq R[t], \quad \mathcal{R}' = \mathcal{R}'(\mathcal{F}) = \sum_{n \in \mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}]$$

and call them *the Rees algebra*, *the extended Rees algebra of  $R$  with respect to  $\mathcal{F}$* , respectively. Here  $t$  stands for an indeterminate over  $R$ .

Let  $M$  be an  $R$ -module,  $\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}$  an  $\mathcal{F}$ -filtration of  $R$ -submodules of  $M$ , that is,  $M_n$  is an  $R$ -submodule of  $M$ ,  $M_n \supseteq M_{n+1}$ ,  $F_m M_n \subseteq M_{m+n}$  for all  $m, n \in \mathbb{Z}$  and  $M_0 = M$ . We set

$$\mathcal{R}(\mathcal{M}) = \sum_{n \geq 0} t^n \otimes M_n \subseteq R[t] \otimes_R M, \quad \mathcal{R}'(\mathcal{M}) = \sum_{n \in \mathbb{Z}} t^n \otimes M_n \subseteq R[t, t^{-1}] \otimes_R M$$

which we call *the Rees module*, *the extended Rees module of  $M$  with respect to  $\mathcal{M}$* , respectively, where

$$t^n \otimes M_n = \{t^n \otimes x \mid x \in M_n\} \subseteq R[t, t^{-1}] \otimes_R M$$

for all  $n \in \mathbb{Z}$ . Then  $\mathcal{R}(\mathcal{M})$  (resp.  $\mathcal{R}'(\mathcal{M})$ ) is a graded module over  $\mathcal{R}$  (resp.  $\mathcal{R}'$ ).

If  $F_1 \neq R$ , then we define *the associated graded ring  $\mathcal{G}$  of  $R$  with respect to  $\mathcal{F}$  and the associated graded module  $\mathcal{G}(\mathcal{M})$  of  $M$  with respect to  $\mathcal{M}$*  as follows.

$$\mathcal{G} = \mathcal{G}(\mathcal{F}) = \mathcal{R}'/u\mathcal{R}', \quad \mathcal{G}(\mathcal{M}) = \mathcal{R}'(\mathcal{M})/u\mathcal{R}'(\mathcal{M}),$$

where  $u = t^{-1}$ . Then  $\mathcal{G}(\mathcal{M})$  is a graded module over  $\mathcal{G}$  and the composite map

$$\psi : \mathcal{R}(\mathcal{M}) \xrightarrow{i} \mathcal{R}'(\mathcal{M}) \xrightarrow{\varepsilon} \mathcal{G}(\mathcal{M})$$

is surjective and  $\text{Ker} \psi = u\mathcal{R}'(\mathcal{M}) \cap \mathcal{R}(\mathcal{M}) = u[\mathcal{R}(\mathcal{M})]_+$ , where  $[\mathcal{R}(\mathcal{M})]_+ = \sum_{n > 0} t^n \otimes M_n$ .

For the rest of this section, we assume that  $F_1 \neq R$ ,  $\mathcal{R} = \mathcal{R}(\mathcal{F})$  is Noetherian and  $\mathcal{R}(\mathcal{M})$  is finitely generated. Then we have the following. The proof of Proposition 2.3 is based on the results [CGT, Proposition 5.1]. Since it plays an important role in this paper, let us give a brief proof for the sake of completeness.

**Proposition 2.3.** *The following assertions hold true.*

(1) Let  $P \in \text{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$ . Then  $\mathfrak{p} \in \text{Ass}_R M$ ,  $P = \mathfrak{p}R[t] \cap \mathcal{R}$  and

$$\dim \mathcal{R}/P = \begin{cases} \dim R/\mathfrak{p} + 1 & \text{if } \dim R/\mathfrak{p} < \infty, F_1 \not\subseteq \mathfrak{p}, \\ \dim R/\mathfrak{p} & \text{otherwise,} \end{cases}$$

where  $\mathfrak{p} = P \cap R$ .

- (2)  $\mathfrak{p}R[t] \cap \mathcal{R} \in \text{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$  for every  $\mathfrak{p} \in \text{Ass}_R M$ .  
 (3) Suppose that  $M \neq (0)$ ,  $d = \dim_R M < \infty$  and there exists  $\mathfrak{p} \in \text{Assh}_R M$  such that  $F_1 \not\subseteq \mathfrak{p}$ . Then  $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d + 1$ .

*Proof.* (1) Let  $P \in \text{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$ . Then  $P \in \text{Ass}_{\mathcal{R}} R[t] \otimes_R M$ , so that  $P = Q \cap \mathcal{R}$  for some

$$Q \in \text{Ass}_{R[t]} R[t] \otimes_R M = \bigcup_{\mathfrak{p} \in \text{Ass}_R M} \text{Ass}_{R[t]} R[t]/\mathfrak{p}R[t].$$

Thus there exists  $\mathfrak{p} \in \text{Ass}_R M$  such that  $\mathfrak{p} = Q \cap R$  and  $Q = \mathfrak{p}R[t]$ . Therefore  $P = \mathfrak{p}R[t] \cap \mathcal{R}$ ,  $\mathfrak{p} = P \cap R$ . Put  $\overline{R} = R/\mathfrak{p}$ . Then  $\overline{\mathcal{F}} = \{F_n \overline{R}\}_{n \in \mathbb{Z}}$  is a filtration of ideals of  $\overline{R}$  and  $\mathcal{R}/P \cong \mathcal{R}(\overline{\mathcal{F}})$  as graded  $R$ -algebras. Hence the assertion holds by [GN, Part II, Lemma (2.2)].

(2) Let  $\mathfrak{p} \in \text{Ass}_R M$ . We write  $\mathfrak{p} = (0) :_R x$  for some  $x \in M$ . Then  $(0) :_{\mathcal{R}} \xi = \mathfrak{p}R[t] \cap \mathcal{R}$  where  $\xi = 1 \otimes x \in [\mathcal{R}(\mathcal{M})]_0$ .

(3) Follows from the assertions (1), (2).  $\square$

**Corollary 2.4.** Suppose that  $R$  is a local ring and  $M \neq (0)$ . Then

$$\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = \begin{cases} \dim_R M + 1 & \text{if there exists } \mathfrak{p} \in \text{Assh}_R M \text{ such that } F_1 \not\subseteq \mathfrak{p}, \\ \dim_R M & \text{otherwise.} \end{cases}$$

Similarly we are able to determine the structure of associated prime ideals of the extended Rees modules.

**Proposition 2.5.** The following assertions hold true.

- (1) Let  $P \in \text{Ass}_{\mathcal{R}'} \mathcal{R}'(\mathcal{M})$ . Then  $\mathfrak{p} \in \text{Ass}_R M$ ,  $P = \mathfrak{p}R[t, t^{-1}] \cap \mathcal{R}'$  and  $\dim \mathcal{R}/P = \dim R/\mathfrak{p} + 1$ , where  $\mathfrak{p} = P \cap R$ .  
 (2)  $\mathfrak{p}R[t, t^{-1}] \cap \mathcal{R}' \in \text{Ass}_{\mathcal{R}'} \mathcal{R}'(\mathcal{M})$  for every  $\mathfrak{p} \in \text{Ass}_R M$ .  
 (3) Suppose that  $M \neq (0)$ . Then  $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{M}) = \dim_R M + 1$ .

Apply Proposition 2.5, we get the following.

**Corollary 2.6.** Suppose  $R$  is a local ring and  $M \neq (0)$ . Then  $\dim_{\mathcal{G}} \mathcal{G}(\mathcal{M}) = \dim_R M$ .

### 3. PROOF OF THEOREM 1.1 AND THEOREM 1.2

This section aims to prove Theorem 1.1 and Theorem 1.2. In what follows, let  $(R, \mathfrak{m})$  be a Noetherian local ring,  $M \neq (0)$  a finitely generated  $R$ -module of dimension  $d$ . Let  $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$  be a filtration of ideals of  $R$  with  $F_1 \neq R$ ,  $\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}$  a  $\mathcal{F}$ -filtration of  $R$ -submodules of  $M$ . We put  $\mathfrak{a} = \mathcal{R}(\mathcal{F})_+ = \sum_{n > 0} F_n t^n$ .

Throughout this section we assume that  $\mathcal{R} = \mathcal{R}(\mathcal{F})$  is a Noetherian ring and  $\mathcal{R}(\mathcal{M})$  is finitely generated. Let  $1 \leq i \leq \ell$ . We set

$$\mathcal{D}_i = \{M_n \cap D_i\}_{n \in \mathbb{Z}}, \quad \mathcal{C}_i = \{[(M_n \cap D_i) + D_{i-1}]/D_{i-1}\}_{n \in \mathbb{Z}}.$$

Then  $\mathcal{D}_i$  (resp.  $\mathcal{C}_i$ ) is a  $\mathcal{F}$ -filtration of  $R$ -submodules of  $D_i$  (resp.  $C_i$ ). Look at the following exact sequence

$$0 \rightarrow [\mathcal{D}_{i-1}]_n \rightarrow [\mathcal{D}_i]_n \rightarrow [\mathcal{C}_i]_n \rightarrow 0$$

of  $R$ -modules for all  $n \in \mathbb{Z}$ . We then have the exact sequences

$$0 \rightarrow \mathcal{R}(\mathcal{D}_{i-1}) \rightarrow \mathcal{R}(\mathcal{D}_i) \rightarrow \mathcal{R}(\mathcal{C}_i) \rightarrow 0$$

$$0 \rightarrow \mathcal{R}'(\mathcal{D}_{i-1}) \rightarrow \mathcal{R}'(\mathcal{D}_i) \rightarrow \mathcal{R}'(\mathcal{C}_i) \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow \mathcal{G}(\mathcal{D}_{i-1}) \rightarrow \mathcal{G}(\mathcal{D}_i) \rightarrow \mathcal{G}(\mathcal{C}_i) \rightarrow 0$$

of graded modules. Since  $\mathcal{R}(\mathcal{D}_i)$  is a finitely generated  $\mathcal{R}$ -module, so is  $\mathcal{R}(\mathcal{C}_i)$ .

**Lemma 3.1.** (cf. [CGT, Proposition 5.1])  *$\{\mathcal{R}'(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $\mathcal{R}'(\mathcal{M})$ . If  $F_1 \not\subseteq \mathfrak{p}$  for every  $\mathfrak{p} \in \text{Ass}_R M$ , then  $\{\mathcal{R}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $\mathcal{R}(\mathcal{M})$ .*

*Proof.* Let  $1 \leq i \leq \ell$ . Then  $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{D}_i) = d_i + 1$ , since  $D_i \neq (0)$ . Let  $P \in \text{Ass}_{\mathcal{R}'} \mathcal{R}'(\mathcal{C}_i)$ . Thanks to Proposition 2.5, we then have  $\dim \mathcal{R}'/P = d_i + 1 = \dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{C}_i)$ . By using [GHS, Theorem 2.3],  $\{\mathcal{R}'(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $\mathcal{R}'(\mathcal{M})$ . Similarly we obtain the last assertion.  $\square$

We now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* The equivalence of conditions (1) and (2) is similar to the proof of Proposition 2.2. Let us make sure of the last assertion. Look at the following exact sequences

$$0 \rightarrow \mathcal{R}'(\mathcal{C}_i) \xrightarrow{\varphi} R[t, t^{-1}] \otimes_R C_i \rightarrow X = \text{Coker} \varphi \rightarrow 0$$

of graded  $\mathcal{R}'$ -modules for  $1 \leq i \leq \ell$ . Since  $\mathcal{R}'(\mathcal{C}_i)$  is a Cohen-Macaulay  $\mathcal{R}'$ -module and  $X_u = (0)$ , we have  $R[t, t^{-1}] \otimes_R C_i$  is Cohen-Macaulay. Therefore  $M$  is a sequentially Cohen-Macaulay  $R$ -module, because  $C_i$  is Cohen-Macaulay.  $\square$

From now on, we focus our attention on the proof of Theorem 1.2. To do this, we need some auxiliaries.

**Lemma 3.2.** *Let  $P \in \text{Spec} \mathcal{R}$  such that  $P \not\supseteq \mathfrak{a}$ . If  $\mathcal{G}(\mathcal{M})_P \neq (0)$  (resp.  $\mathcal{R}(\mathcal{M})_P \neq (0)$  and  $P \supseteq u\mathfrak{a}$ ), then  $\mathcal{R}(\mathcal{M})_P \neq (0)$  (resp.  $\mathcal{G}(\mathcal{M})_P \neq (0)$ ). When this is the case, the following assertions hold true.*

- (1)  $\mathcal{R}(\mathcal{M})_P$  is a Cohen-Macaulay  $\mathcal{R}_P$ -module if and only if  $\mathcal{G}(\mathcal{M})_P$  is a Cohen-Macaulay  $\mathcal{G}_P$ -module.
- (2)  $\dim_{\mathcal{R}_P} \mathcal{R}(\mathcal{M})_P = \dim_{\mathcal{R}_P} \mathcal{G}(\mathcal{R})_P + 1$ .

*Proof.* Let  $P \in \text{Spec} \mathcal{R}$  such that  $P \not\supseteq \mathfrak{a}$ , but  $P \supseteq u\mathfrak{a}$ . We choose a homogeneous element  $\xi = at^n \in \mathfrak{a} \setminus P$  where  $n > 0$ ,  $a \in F_n$ . Then we get  $x = u\xi = at^{n-1} \in P$ , since  $P \supseteq u\mathfrak{a}$ .

**Claim 3.3.** *If  $Q \in \text{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$  such that  $Q \subseteq P$ , then  $x \notin Q$ . Therefore  $x$  is a non-zerodivisor on  $\mathcal{R}(\mathcal{M})_P$ .*

*Proof of Claim 3.3.* We assume that there exists  $Q \in \text{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$  such that  $Q \subseteq P$ , but  $x \in Q$ . Write  $Q = (0) :_{\mathcal{R}} \eta$  where  $\eta = t^\ell \otimes m$  ( $\ell \in \mathbb{Z}$ ,  $m \in M_\ell$ ). Then we have  $\xi = at^n \in (0) :_{\mathcal{R}} \eta = Q \subseteq P$ , which implies a contradiction.  $\square$



Since  $P \not\subseteq \mathfrak{a}$ , we get  $\mathcal{R}_P = \mathcal{R}'_P$  and  $\mathcal{R}(\mathcal{M})_P = \mathcal{R}'(\mathcal{M})_P$ . Therefore

$$(u\mathfrak{a})\mathcal{R}_P = (u\mathfrak{a})\mathcal{R}'_P = u\mathcal{R}'_P = x\mathcal{R}'_P \quad \text{and} \quad (u\mathfrak{a})\mathcal{R}(\mathcal{M}) \subseteq u[\mathcal{R}(\mathcal{M})]_+.$$

Hence  $[u\mathcal{R}(\mathcal{M})_+]_P = x\mathcal{R}'(\mathcal{M})_P = x\mathcal{R}(\mathcal{M})_P$ , so that

$$\mathcal{R}(\mathcal{M})_P/x\mathcal{R}(\mathcal{M})_P \cong \mathcal{G}(\mathcal{M})_P$$

as  $\mathcal{R}_P$ -modules. On the other hand, let  $P \in \text{Spec } \mathcal{R}$  such that  $\mathcal{G}(\mathcal{M})_P \neq (0)$ . Then  $P \supseteq u\mathfrak{a}$ , since  $u\mathfrak{a} = u\mathcal{R}' \cap \mathcal{R} = \text{Ker}(\mathcal{R} \xrightarrow{i} \mathcal{R}' \xrightarrow{\varepsilon} \mathcal{G})$ . Therefore the assertions immediately come from the above isomorphism.  $\square$

Here we need the following fact, which was originally given by G. Faltings.

**Fact 3.4** ([F]). Let  $I$  be an ideal of  $R$  and  $t \in \mathbb{Z}$ . Consider the following two conditions.

- (1) There exists an integer  $\ell > 0$  such that  $I^\ell \cdot H_m^i(M) = (0)$  for each  $i \neq t$ .
- (2)  $M_{\mathfrak{p}}$  is a Cohen Macaulay  $R_{\mathfrak{p}}$ -module and  $t = \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p}$  for every  $\mathfrak{p} \in \text{Supp}_R M$  but  $\mathfrak{p} \not\subseteq I$ .

Then the implication (1)  $\Rightarrow$  (2) holds true. The converse holds, if  $R$  is a homomorphic image of a Gorenstein local ring.

Let  $\mathfrak{M}$  be a unique graded maximal ideal of  $\mathcal{R}$ . Although a part of the proof of Proposition 3.5 is due to the result [TI], we note the brief proof for the sake of completeness.

**Proposition 3.5.** *Suppose that  $H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M}))$  is a finitely graded  $\mathcal{R}$ -module for all  $i \neq d$ . Then  $H_{\mathfrak{M}}^i(\mathcal{R}(\mathcal{M}))$  is a finitely graded  $\mathcal{R}$ -module for all  $i \neq d+1$ .*

*Proof.* Passing to the completion and taking the local duality theorem, it is enough to show that there exists an integer  $\ell > 0$  such that  $\mathfrak{a}^\ell \cdot H_{\mathfrak{M}}^i(\mathcal{R}(M)) = (0)$  for every  $i \neq d+1$ . To see this, let  $P \in \text{Supp}_{\mathcal{R}} \mathcal{R}(M)$  such that  $P \not\subseteq \mathfrak{a}$  and  $P \subseteq \mathfrak{M}$ . Put  $L = u\mathfrak{a} = u\mathcal{R}' \cap \mathcal{R}$ .

**Claim 3.6.**  $\sqrt{P^* + L} \not\subseteq \mathfrak{a}$ .

*Proof of Claim 3.6.* Suppose that  $P^* + L \supseteq \mathfrak{a}^\ell$  for some  $\ell > 0$ . Since  $\mathcal{R}/\mathfrak{a}^\ell$  is finitely graded, we can choose an integer  $s > 0$  such that  $[\mathcal{R}/\mathfrak{a}^\ell]_n = (0)$  for all  $n \geq s$ . Then

$$\mathcal{R}_n = F_n t^n \subseteq [P^*]_n + F_{n+1} t^n$$

for all  $n \geq s$ . On the other hand, for each  $n \geq 0$ , we set

$$I_n = \{a \in R \mid at^n \in P^*\}.$$

Then  $I_n$  is an ideal of  $R$  and  $I_n \subseteq F_n$  and  $I_n \supseteq I_{n+1}$  for all  $n \geq 0$ . Hence  $F_n \subseteq I_n + F_k$  for all  $n \geq s$ ,  $k \in \mathbb{Z}$ . Since  $\mathcal{R}$  is Noetherian, we get  $\mathcal{R}^{(d)} = R[F_d t^d]$  for some  $d > 0$ , so that  $F_{d\ell} = (F_d)^\ell$  for all  $\ell > 0$ . We then have

$$F_n \subseteq \bigcap_{\ell > 0} [I_n + (F_d)^\ell] = I_n$$

for all  $n \geq s$ , whence  $\mathcal{R}_n \subseteq P^*$ . Thus

$$\mathfrak{a}^s \subseteq \sum_{n \geq s} \mathcal{R}_n \subseteq P^* \subseteq P.$$

which is impossible, because  $\mathfrak{a} \not\subseteq P$ .  $\square$

Therefore we can take  $Q \in \text{Min}_{\mathcal{R}} \mathcal{R}/[P^* + L]$  such that  $\mathfrak{a} \not\subseteq Q \subseteq \mathfrak{M}$ . Then  $\mathcal{R}(\mathcal{M})_Q \neq (0)$ , because  $\mathcal{R}(\mathcal{M})_{P^*} \neq (0)$  and  $P^* \subseteq Q$ . Thanks to Lemma 3.2,  $\mathcal{G}(\mathcal{M})_Q \neq (0)$ . Then  $\mathcal{G}(\mathcal{M})_Q$  is Cohen-Macaulay and  $\dim_{\mathcal{R}_Q} \mathcal{G}(\mathcal{M})_Q + \dim \mathcal{R}_{\mathfrak{M}}/Q\mathcal{R}_{\mathfrak{M}} = d$  by using Fact 3.4. Hence  $\mathcal{R}(\mathcal{M})_Q$  is Cohen-Macaulay and  $\dim_{\mathcal{R}_Q} \mathcal{R}(\mathcal{M})_Q + \dim \mathcal{R}_{\mathfrak{M}}/Q\mathcal{R}_{\mathfrak{M}} = d + 1$  by Lemma 3.2.

Since  $P^* \subseteq Q$ ,  $\mathcal{R}(\mathcal{M})_{P^*}$  is Cohen-Macaulay, so is  $\mathcal{R}(\mathcal{M})_P$ . We also have

$$\begin{aligned} d + 1 &= \dim_{\mathcal{R}_Q} \mathcal{R}(\mathcal{M})_Q + \dim \mathcal{R}_{\mathfrak{M}}/Q\mathcal{R}_{\mathfrak{M}} \\ &= (\dim_{\mathcal{R}_{P^*}} \mathcal{R}(\mathcal{M})_{P^*} + \dim \mathcal{R}_Q/P^*\mathcal{R}_Q) + (\dim \mathcal{R}_{\mathfrak{M}}/P^*\mathcal{R}_{\mathfrak{M}} - \dim \mathcal{R}_Q/P^*\mathcal{R}_Q) \\ &= \dim_{\mathcal{R}_{P^*}} \mathcal{R}(\mathcal{M})_{P^*} + \dim \mathcal{R}_{\mathfrak{M}}/P^*\mathcal{R}_{\mathfrak{M}} \\ &= \dim_{\mathcal{R}_P} \mathcal{R}(\mathcal{M})_P + \dim \mathcal{R}_{\mathfrak{M}}/P\mathcal{R}_{\mathfrak{M}}. \end{aligned}$$

Thanks to Fact 3.4 again, there exists  $\ell > 0$  such that

$$\mathfrak{a}^\ell \cdot H_{\mathfrak{M}}^i(\mathcal{R}(\mathcal{M})) = (0) \text{ for each } i \neq d + 1$$

which shows  $H_{\mathfrak{M}}^i(\mathcal{R}(\mathcal{M}))$  is finitely graded.  $\square$

We set

$$a(N) = \max\{n \in \mathbb{Z} \mid [H_{\mathfrak{M}}^t(N)]_n \neq (0)\}$$

for a finitely generated graded  $\mathcal{R}$ -module  $N$  of dimension  $t$ , and call it *the  $a$ -invariant of  $N$*  (see [GW, DEFINITION (3.1.4)]). With this notation we have the following.

**Lemma 3.7.** *The following assertions hold true.*

- (1)  $[H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M}))]_n = (0)$  for all  $n \geq 0$ .
  - (2) If  $[H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M}))]_{-1} = (0)$ , then  $H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M})) = (0)$ .
- Consequently  $a(\mathcal{R}(\mathcal{M})) = -1$ , if  $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d + 1$ .

*Proof.* We look at the following exact sequences

$$0 \rightarrow L \rightarrow \mathcal{R}(\mathcal{M}) \rightarrow M \rightarrow 0$$

$$0 \rightarrow L(1) \rightarrow \mathcal{R}(\mathcal{M}) \rightarrow \mathcal{G}(\mathcal{M}) \rightarrow 0$$

of graded  $\mathcal{R}$ -modules, where  $L = \mathcal{R}(\mathcal{M})_+$ . By applying the local cohomology functors to the above sequences, we get

$$H_{\mathfrak{m}}^d(M) \rightarrow H_{\mathfrak{M}}^{d+1}(L) \rightarrow H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M})) \rightarrow 0$$

and

$$H_{\mathfrak{M}}^d(\mathcal{G}(\mathcal{M})) \rightarrow H_{\mathfrak{M}}^{d+1}(L)(1) \rightarrow H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M})) \rightarrow 0.$$

Thus

$$\begin{aligned} [H_{\mathfrak{M}}^{d+1}(L)]_n &\cong [H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M}))]_n \text{ for } n \neq 0, \text{ and} \\ [H_{\mathfrak{M}}^{d+1}(L)]_{n+1} &\rightarrow [H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M}))]_n \rightarrow 0 \text{ for } n \in \mathbb{Z}. \end{aligned}$$

Therefore  $[H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M}))]_n = (0)$  for  $n \geq 0$ , because  $H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M}))$  is Artinian. Moreover we have

$$[H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M}))]_{-1} \rightarrow [H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M}))]_n \rightarrow 0$$

for  $n < 0$ , so we get the assertion (2).  $\square$



We finally arrive at the following Theorem 3.8 which is a module version of the results [GN, Part II, Theorem (1.1)], [V, Theorem 1.1] (see also [TI, Theorem 1.1], [GS, Theorem (1.1)]).

**Theorem 3.8.** *The following conditions are equivalent.*

- (1)  $\mathcal{R}(\mathcal{M})$  is a Cohen-Macaulay  $\mathcal{R}$ -module and  $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d + 1$ .
- (2)  $H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M})) = [H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M}))]_{-1}$  for every  $i < d$  and  $a(\mathcal{G}(\mathcal{M})) < 0$ .

When this is the case,  $[H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M}))]_{-1} \cong H_{\mathfrak{m}}^i(M)$  as  $R$ -modules for all  $i < d$ .

*Proof.* Consider the following exact sequences

$$\begin{aligned}
 (*) \quad & \cdots \rightarrow H_{\mathfrak{m}}^i(L) \rightarrow H_{\mathfrak{M}}^i(\mathcal{R}(\mathcal{M})) \rightarrow H_{\mathfrak{m}}^i(M) \rightarrow H_{\mathfrak{M}}^{i+1}(L) \rightarrow H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M})) \rightarrow \cdots \\
 (**) \quad & \cdots \rightarrow H_{\mathfrak{m}}^i(L)(1) \rightarrow H_{\mathfrak{M}}^i(\mathcal{R}(\mathcal{M})) \rightarrow H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M})) \rightarrow H_{\mathfrak{M}}^{i+1}(L)(1) \rightarrow H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M})) \rightarrow \cdots
 \end{aligned}$$

for each  $i < d$ .

Firstly we assume that  $\mathcal{R}(\mathcal{M})$  is a Cohen-Macaulay  $\mathcal{R}$ -module of dimension  $d + 1$ . Then

$$H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{M}}^{i+1}(L) \quad \text{and} \quad H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M})) \cong H_{\mathfrak{M}}^{i+1}(L)(1)$$

for  $i < d$ . Therefore we get  $H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M})) = [H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M}))]_{-1}$  and  $[H_{\mathfrak{M}}^i(\mathcal{G}(\mathcal{M}))]_{-1} \cong H_{\mathfrak{m}}^i(M)$  as  $R$ -modules. Since  $\mathcal{R}(\mathcal{M})$  is Cohen-Macaulay, we have

$$\begin{aligned}
 0 \rightarrow H_{\mathfrak{m}}^d(M) \rightarrow H_{\mathfrak{M}}^{d+1}(L) \rightarrow H_{\mathfrak{M}}^{d+1}(\mathcal{R}(\mathcal{M})) \rightarrow 0 \\
 0 \rightarrow H_{\mathfrak{M}}^d(\mathcal{G}(\mathcal{M})) \rightarrow H_{\mathfrak{M}}^{d+1}(L)(1).
 \end{aligned}$$

Therefore  $a(\mathcal{G}(\mathcal{M})) < 0$  by using Lemma 3.7.

Conversely, let  $i < d$ . Thanks to the above sequences (\*), (\*\*) and our hypothesis, we get

$$\begin{aligned}
 [H_{\mathfrak{M}}^{i+1}(L)]_{n+1} &\cong [H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M}))]_n \\
 [H_{\mathfrak{M}}^{i+1}(L)]_{n+1} &\cong [H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M}))]_{n+1}
 \end{aligned}$$

for each  $n \geq 0$ . Hence  $[H_{\mathfrak{M}}^i(\mathcal{R}(\mathcal{M}))]_n = (0)$  for  $n \geq 0$ , since  $H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M}))$  is Artinian. Moreover, we then have

$$0 \rightarrow [H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M}))]_n \rightarrow [H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M}))]_{n-1}.$$

for  $n < 0$  by above sequences (\*) and (\*\*). Thanks to Proposition 3.5,  $H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M}))$  is a finitely graded  $\mathcal{R}$ -module for  $i < d$ . Whence  $[H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M}))]_n = (0)$ , which shows  $H_{\mathfrak{M}}^{i+1}(\mathcal{R}(\mathcal{M})) = (0)$  for all  $i < d$ . Hence  $\mathcal{R}(\mathcal{M})$  is a Cohen-Macaulay  $\mathcal{R}$ -module of dimension  $d + 1$ .  $\square$

**Corollary 3.9.** *Suppose that  $M$  is a Cohen-Macaulay  $R$ -module. Then the following conditions are equivalent.*

- (1)  $\mathcal{R}(\mathcal{M})$  is a Cohen-Macaulay  $\mathcal{R}$ -module and  $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d + 1$ .
- (2)  $\mathcal{G}(\mathcal{M})$  is a Cohen-Macaulay  $\mathcal{G}$ -module and  $a(\mathcal{G}(\mathcal{M})) < 0$ .

We now reach the goal of this section.

*Proof of Theorem 1.2.* Thanks to Lemma 3.1,  $\mathcal{R}(\mathcal{M})$  is a sequentially Cohen-Macaulay  $\mathcal{R}$ -module if and only if  $\mathcal{R}(\mathcal{C}_i)$  is Cohen-Macaulay for every  $1 \leq i \leq \ell$ . The latter condition is equivalent to saying that  $\mathcal{G}(\mathcal{C}_i)$  is a Cohen-Macaulay  $\mathcal{G}$ -module and  $a(\mathcal{G}(\mathcal{C}_i)) < 0$  for all  $1 \leq i \leq \ell$  by Corollary 3.9. Hence we get the equivalence between (1) and (2).  $\square$

We close this section by stating the ring version of Theorem 1.1 and Theorem 1.2. Let  $(R, \mathfrak{m})$  be a Noetherian local ring,  $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$  a filtration of ideals of  $R$  such that  $F_1 \neq R$ . We assume that  $\mathcal{R} = \mathcal{R}(\mathcal{F})$  is a Noetherian ring. Let  $\{D_i\}_{0 \leq i \leq \ell}$  be the dimension filtration of  $R$ . Then  $\mathcal{D}_i = \{F_n \cap D_i\}_{n \in \mathbb{Z}}$  (resp.  $\mathcal{C}_i = \{[F_n \cap D_i + D_{i-1}]/D_{i-1}\}_{n \in \mathbb{Z}}$ ) is a  $\mathcal{F}$ -filtration of  $D_i$  (resp.  $C_i$ ) for all  $1 \leq i \leq \ell$ .

**Theorem 3.10.** *The following conditions are equivalent.*

- (1)  $\mathcal{R}'$  is a sequentially Cohen-Macaulay ring.
- (2)  $\mathcal{G}$  is a sequentially Cohen-Macaulay ring and  $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $\mathcal{G}$ .

When this is the case,  $R$  is a sequentially Cohen-Macaulay ring.

**Theorem 3.11.** *Suppose that  $R$  is a sequentially Cohen-Macaulay ring and  $F_1 \not\subseteq \mathfrak{p}$  for every  $\mathfrak{p} \in \text{Ass} R$ . Then the following conditions are equivalent.*

- (1)  $\mathcal{R}$  is a sequentially Cohen-Macaulay ring.
- (2)  $\mathcal{G}$  is a sequentially Cohen-Macaulay ring,  $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $\mathcal{G}$  and  $a(\mathcal{G}(\mathcal{C}_i)) < 0$  for all  $1 \leq i \leq \ell$ .

When this is the case,  $\mathcal{R}'$  is a sequentially Cohen-Macaulay ring.

#### 4. SEQUENTIALLY COHEN-MACAULAY PROPERTY IN $E^\natural$

In this section let  $R = \sum_{n \geq 0} R_n$  be a  $\mathbb{Z}$ -graded ring. We put  $F_n = \sum_{k \geq n} R_k$  for all  $n \in \mathbb{Z}$ . Then  $F_n$  is a graded ideal of  $R$ ,  $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$  is a filtration of ideals of  $R$  and  $F_1 := R_+ \neq R$ . Let  $E$  be a graded  $R$ -module with  $E_n = (0)$  for all  $n < 0$ . Put  $E_{(n)} = \sum_{k \geq n} E_k$  for all  $n \in \mathbb{Z}$ . Then  $E_{(n)}$  is a graded  $R$ -submodule of  $E$ ,  $\mathcal{E} = \{E_{(n)}\}_{n \in \mathbb{Z}}$  is an  $\mathcal{F}$ -filtration of  $R$ -submodules of  $E$  and  $E_{(0)} = E$ . Then we have  $R = \mathcal{G}(\mathcal{F})$  and  $E = \mathcal{G}(\mathcal{E})$ . Set  $R^\natural := \mathcal{R}(\mathcal{F})$  and  $E^\natural := \mathcal{R}(\mathcal{E})$ .

Suppose that  $R$  is a Noetherian ring and  $E \neq (0)$  is a finitely generated graded  $R$ -module with  $d = \dim_R E < \infty$ . Notice that  $R^\natural$  is Noetherian and  $E^\natural, \mathcal{R}'(\mathcal{E})$  are finitely generated.

We note the following.

**Lemma 4.1.** *The following assertions hold true.*

- (1)  $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{E}) = d + 1$ .
- (2) Suppose that there exists  $\mathfrak{p} \in \text{Assh}_R E$  such that  $F_1 \not\subseteq \mathfrak{p}$ . Then  $\dim_{R^\natural} E^\natural = d + 1$ .

*Proof.* See Proposition 2.3, Proposition 2.5. □

Let  $D_0 \subsetneq D_1 \subsetneq \dots \subsetneq D_\ell = E$  be the dimension filtration of  $E$ . We set  $C_i = D_i/D_{i-1}$ ,  $d_i = \dim_R D_i$  for every  $1 \leq i \leq \ell$ . Then  $D_i$  is a graded  $R$ -submodule of  $E$  for all  $0 \leq i \leq \ell$ . Let  $1 \leq i \leq \ell$ . Then from the exact sequence

$$0 \rightarrow [D_{i-1}]_{(n)} \rightarrow [D_i]_{(n)} \rightarrow [C_i]_{(n)} \rightarrow 0$$

of graded  $R$ -modules for all  $n \in \mathbb{Z}$ , we get the exact sequences

$$0 \rightarrow \mathcal{R}(\mathcal{D}_{i-1}) \rightarrow \mathcal{R}(\mathcal{D}_i) \rightarrow \mathcal{R}(\mathcal{C}_i) \rightarrow 0$$

$$0 \rightarrow \mathcal{R}'(\mathcal{D}_{i-1}) \rightarrow \mathcal{R}'(\mathcal{D}_i) \rightarrow \mathcal{R}'(\mathcal{C}_i) \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow \mathcal{G}(\mathcal{D}_{i-1}) \rightarrow \mathcal{G}(\mathcal{D}_i) \rightarrow \mathcal{G}(\mathcal{C}_i) \rightarrow 0$$

of graded modules, where  $\mathcal{D}_i = \{[D_i]_{(n)}\}_{n \in \mathbb{Z}}$ ,  $\mathcal{C}_i = \{[C_i]_{(n)}\}_{n \in \mathbb{Z}}$ . By the same technique as in the proof of Lemma 3.1, we obtain the dimension filtration of  $\mathcal{R}'(E)$  and  $E^\natural$  as follows.

**Lemma 4.2.**  *$\{\mathcal{R}'(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $\mathcal{R}'(\mathcal{E})$ . If  $F_1 \not\subseteq \mathfrak{p}$  for every  $\mathfrak{p} \in \text{Ass}_R E$ , then  $\{\mathcal{R}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $E^\natural$ .*

Hence we get the following, which characterize the sequentially Cohen-Macaulayness of  $\mathcal{R}'(\mathcal{E})$ .

**Proposition 4.3.** *The following conditions are equivalent.*

- (1)  $\mathcal{R}'(\mathcal{E})$  is a sequentially Cohen-Macaulay  $\mathcal{R}'$ -module.
- (2)  $E$  is a sequentially Cohen-Macaulay  $R$ -module.

*Proof.* (1)  $\Rightarrow$  (2) Follows from the fact that  $C_i = \mathcal{G}(\mathcal{C}_i)$  for each  $1 \leq i \leq \ell$ .

(2)  $\Rightarrow$  (1) We get  $\mathcal{G}(\mathcal{C}_i)$  is Cohen-Macaulay for all  $1 \leq i \leq \ell$ . Let  $Q \in \text{Supp}_{\mathcal{R}'} \mathcal{R}'(\mathcal{C}_i)$ . We may assume  $u \notin Q$ . Then  $\mathcal{R}'(\mathcal{C}_i)_u = R[t, t^{-1}] \otimes_R C_i$  is Cohen-Macaulay since  $C_i$  is Cohen-Macaulay. Hence  $\mathcal{R}'(\mathcal{C}_i)_Q$  is a Cohen-Macaulay  $\mathcal{R}'_Q$ -module.  $\square$

Now we study the question of when  $E^\natural$  is sequentially Cohen-Macaulay. The key is the following.

**Lemma 4.4.** *Suppose  $R_0$  is a local ring,  $E$  is a Cohen-Macaulay  $R$ -module and  $F_1 \not\subseteq \mathfrak{p}$  for some  $\mathfrak{p} \in \text{Assh}_R E$ . Then the following conditions are equivalent.*

- (1)  $E^\natural$  is a Cohen-Macaulay  $R^\natural$ -module.
- (2)  $a(E) < 0$ .

*Proof.* Let  $P = \mathfrak{m}R + R_+$ , where  $\mathfrak{m}$  denotes the maximal ideal of  $R_0$ . Then  $P \supseteq F_1$ . Since  $R_+(E_{(n)}/E_{(n+1)}) = (0)$ ,  $R_+(F_n/F_{n+1}) = (0)$  for all  $n \in \mathbb{Z}$ , we have

$$E = \mathcal{G}(\mathcal{E}) \cong \mathcal{G}(\mathcal{E}_P), \quad R = \mathcal{G}(\mathcal{F}) \cong \mathcal{G}(\mathcal{F}_P).$$

Suppose that  $E^\natural$  is a Cohen-Macaulay  $R^\natural$ -module. Then  $\mathcal{R}(\mathcal{E}_P)$  is Cohen-Macaulay and  $\dim_{\mathcal{R}(R_P)} \mathcal{R}(\mathcal{E}_P) = d + 1$ , whence  $\mathcal{G}(\mathcal{E}_P)$  is Cohen-Macaulay and  $a(\mathcal{G}(\mathcal{E}_P)) < 0$ . Therefore we get  $a(E) < 0$ .

On the other hand, suppose that  $a(E) < 0$ . Then  $\mathcal{R}(\mathcal{E}_P)$  is a Cohen-Macaulay  $\mathcal{R}(R_P)$ -module of dimension  $d + 1$ . Thus  $\mathcal{R}(\mathcal{E})_P$  is Cohen-Macaulay. Now we regard  $\mathcal{R}$  as a  $\mathbb{Z}^2$ -graded ring with the  $\mathbb{Z}^2$ -grading as follows:

$$\mathcal{R}_{(i,j)} = \begin{cases} R_i t^j & i \geq j \geq 0 \\ (0) & \text{otherwise.} \end{cases}$$

Moreover we set

$$\mathcal{R}(\mathcal{E})_{(i,j)} = \begin{cases} t^j \otimes E_i & i \geq j \geq 0 \\ (0) & \text{otherwise,} \end{cases}$$

where  $t^j \otimes E_i = \{t^j \otimes x \mid x \in E_i\}$ . Then  $\mathcal{R}(\mathcal{E})$  is a  $\mathbb{Z}^2$ -graded  $\mathcal{R}$ -module with the above grading  $\mathcal{R}(\mathcal{E})_{(i,j)}$ . Notice that  $\mathcal{R}_{(0,0)} = R_0$  is a local ring, so that  $\mathcal{R}$  is  $H$ -local. Let  $L$  be the  $H$ -maximal ideal of the  $\mathbb{Z}^2$ -graded ring  $\mathcal{R}$ . Then we get  $P \subseteq L$ , whence  $L \cap R = P$ . Therefore  $\mathcal{R}(\mathcal{E})_L$  is a Cohen-Macaulay  $\mathcal{R}_L$ -module, so that  $E^\natural$  is Cohen-Macaulay.  $\square$

Our answer is the following.

**Theorem 4.5.** *Suppose that  $R_0$  is a local ring,  $E$  is a sequentially Cohen-Macaulay  $R$ -module and  $F_1 \not\subseteq \mathfrak{p}$  for every  $\mathfrak{p} \in \text{Ass}_R E$ . Then the following conditions are equivalent.*

- (1)  $E^\natural$  is a sequentially Cohen-Macaulay  $R^\natural$ -module.
- (2)  $a(C_i) < 0$  for all  $1 \leq i \leq \ell$ .

### 5. APPLICATION –STANLEY-REISNER ALGEBRAS–

In this section, let  $V = \{1, 2, \dots, n\}$  ( $n > 0$ ) be a vertex set,  $\Delta$  a simplicial complex on  $V$  such that  $\Delta \neq \emptyset$ . We denote  $\mathcal{F}(\Delta)$  a set of facets of  $\Delta$  and  $m = \sharp\mathcal{F}(\Delta)$  ( $> 0$ ) its cardinality. Let  $S = k[X_1, X_2, \dots, X_n]$  a polynomial ring over a field  $k$ ,  $R = k[\Delta] = S/I_\Delta$  the Stanley-Reisner ring of  $\Delta$  of dimension  $d$ , where  $I_\Delta = (X_{i_1}X_{i_2} \cdots X_{i_r} \mid \{i_1 < i_2 < \cdots < i_r\} \notin \Delta)$  the Stanley-Reisner ideal of  $R$ .

We consider the Stanley-Reisner ring  $R = \sum_{n \geq 0} R_n$  as a  $\mathbb{Z}$ -graded ring and put

$$I_n = \sum_{k \geq n} R_k = \mathfrak{m}^n \text{ for all } n \in \mathbb{Z}$$

where  $\mathfrak{m} = R_+ = \sum_{n > 0} R_n$  is a graded maximal ideal of  $R$ . Then  $\mathcal{I} = \{I_n\}_{n \in \mathbb{Z}}$  is a  $\mathfrak{m}$ -adic filtration of  $R$  and  $I_1 := R_+ \neq R$ .

If  $\Delta$  is shellable, then  $R$  is a sequentially Cohen-Macaulay ring, so by Proposition 4.3 we get the following.

**Proposition 5.1.** *If  $\Delta$  is shellable, then  $\mathcal{R}'(\mathfrak{m})$  is a sequentially Cohen-Macaulay ring.*

Notice that  $\mathfrak{p} \not\supseteq I_1$  for every  $\mathfrak{p} \in \text{Ass} R$  if and only if  $F \neq \emptyset$  for all  $F \in \mathcal{F}(\Delta)$ , which is equivalent to saying that  $\Delta \neq \{\emptyset\}$ .

The goal of this section is the following. Here  $|F_i|$  denotes the cardinality of  $F_i$ .

**Theorem 5.2.** *Suppose that  $\Delta$  is shellable with shelling order  $F_1, F_2, \dots, F_m \in \mathcal{F}(\Delta)$  such that  $\dim F_1 \geq \dim F_2 \geq \cdots \geq \dim F_m$  and  $\Delta \neq \{\emptyset\}$ . Then the following conditions are equivalent.*

- (1)  $\mathcal{R}(\mathfrak{m})$  is a sequentially Cohen-Macaulay ring.
- (2)  $m = 1$  or if  $m \geq 2$ , then  $|F_i| > \sharp\mathcal{F}(\Delta_1 \cap \Delta_2)$  for every  $2 \leq i \leq m$ , where  $\Delta_1 = \langle F_1, F_2, \dots, F_{i-1} \rangle$ ,  $\Delta_2 = \langle F_i \rangle$ .

*Proof.* Thanks to Theorem 4.5,  $\mathcal{R}$  is sequentially Cohen-Macaulay if and only if  $a(C_i) < 0$  for all  $1 \leq i \leq \ell$ , where  $\{D_i\}_{0 \leq i \leq \ell}$  is the dimension filtration of  $R$ ,  $C_i = D_i/D_{i-1}$  and  $d_i = \dim_R D_i$  for all  $1 \leq i \leq \ell$ . If  $m = 1$ , then  $R = k[\Delta] \cong k[X_i \mid i \in F_1]$ , which is a polynomial ring, so that  $\ell = 1$  and  $a(R) = -|F_1| < 0$ . Hence  $\mathcal{R}$  is a Cohen-Macaulay ring by Lemma 4.4.

Suppose that  $m > 1$  and the assertion holds for  $m - 1$ . We put  $\Delta_1 = \langle F_1, F_2, \dots, F_{m-1} \rangle$  and  $\Delta_2 = \langle F_m \rangle$ . If  $\ell = 1$ , then  $\Delta$  is pure. Look at the following exact sequence

$$0 \rightarrow S/I_\Delta \rightarrow S/I_{\Delta_1} \oplus S/I_{\Delta_2} \rightarrow S/I_{\Delta_1} + I_{\Delta_2} \rightarrow 0$$

of graded  $R$ -modules. We then have

$$S/I_{\Delta_1} + I_{\Delta_2} \cong k[\Delta_2]/(\bar{\xi})$$

for some monomials  $\xi \in I_{\Delta_1} \setminus I_{\Delta_2}$  in  $X_1, X_2, \dots, X_n$  with  $0 < \deg \xi = \sharp\mathcal{F}(\Delta_1 \cap \Delta_2)$ . Therefore  $a(S/I_{\Delta_1} + I_{\Delta_2}) = \sharp\mathcal{F}(\Delta_1 \cap \Delta_2) - |F_m|$ . We put  $\mathfrak{m} = R_+$ . Then we have the exact sequence of local cohomology modules as follows

$$0 \rightarrow H_{\mathfrak{m}}^{d-1}(S/I_{\Delta_1} + I_{\Delta_2}) \rightarrow H_{\mathfrak{m}}^d(S/I_\Delta) \rightarrow H_{\mathfrak{m}}^d(S/I_{\Delta_1}) \oplus H_{\mathfrak{m}}^d(S/I_{\Delta_2}) \rightarrow 0.$$

Thus  $a(R) = \max\{\sharp\mathcal{F}(\Delta_1 \cap \Delta_2) - |F_m|, a(k[\Delta_1]), a(k[\Delta_2])\}$ . Hence  $\mathcal{R}$  is sequentially Cohen-Macaulay if and only if  $\sharp\mathcal{F}(\Delta_1 \cap \Delta_2) < |F_m|$  and  $a(k[\Delta_1]) < 0$ . By using the induction arguments, we get the equivalence between (1) and (2).

Suppose now that  $\ell > 1$ . Consider the following exact sequence

$$0 \rightarrow I_{\Delta_1}/I_{\Delta} \rightarrow S/I_{\Delta} \rightarrow S/I_{\Delta_1} \rightarrow 0$$

of graded  $R$ -modules. Then we have

$$I_{\Delta_1}/I_{\Delta} \cong I_{\Delta_1} + I_{\Delta_2}/I_{\Delta_2} = I_{\Delta_1 \cap \Delta_2}/I_{\Delta_2} = (\bar{\xi})$$

where  $\xi \in I_{\Delta_1} \setminus I_{\Delta_2}$  is a homogeneous element with  $0 < \deg \xi = \sharp\mathcal{F}(\Delta_1 \cap \Delta_2) =: t$ . Therefore  $I_{\Delta_1}/I_{\Delta} \cong S/I_{\Delta_2}(-t)$ , so that

$$0 \rightarrow S/I_{\Delta_2}(-t) \xrightarrow{\sigma} S/I_{\Delta} \xrightarrow{\varepsilon} S/I_{\Delta_1} \rightarrow 0.$$

We put  $L = \text{Im} \sigma$ . Then  $L \neq (0)$ ,  $\dim_R L = d_1$  and  $a(L) = t - |F_m|$ . We notice here that  $L \subseteq D_1$ . Now we set  $D_i' = \varepsilon(D_i)$  for every  $1 \leq i \leq \ell$ . Then  $D_1' \subsetneq D_2' \subsetneq \dots \subsetneq D_{\ell}' = k[\Delta_1]$  and  $C_i' := D_i'/D_{i-1}' \cong C_i$  for all  $2 \leq i \leq \ell$ . Hence  $a(C_i) = a(C_i')$  for  $2 \leq i \leq \ell$ .

**Case 1**  $L \subsetneq D_1$  (i.e.,  $D_1' \neq (0)$ )

In this case  $D_0' := (0) \subsetneq D_1' \subsetneq D_2' \subsetneq \dots \subsetneq D_{\ell}' = k[\Delta_1]$  is the dimension filtration of  $k[\Delta_1]$ . Look at the following exact sequence

$$0 \rightarrow L \rightarrow D_1 \rightarrow D_1' \rightarrow 0$$

of  $R$ -modules. Then  $a(D_1) = \max\{a(L), a(D_1')\}$ .

**Case 2**  $L = D_1$  (i.e.,  $D_1' = (0)$ )

Similarly  $(0) = D_1' \subsetneq D_2' \subsetneq \dots \subsetneq D_{\ell}' = k[\Delta_1]$  is the dimension filtration of  $k[\Delta_1]$ .

Summing up, in any case  $\mathcal{R}$  is a sequentially Cohen-Macaulay ring if and only if  $a(L) < 0$  and the assertion (1) holds for the ring  $k[\Delta_1]$ . Hence we get the equivalence of conditions (1) and (2) by using the induction hypothesis.  $\square$

**Remark 5.3.** If  $\Delta$  is shellable, then we can take a shelling order  $F_1, F_2, \dots, F_m \in \mathcal{F}(\Delta)$  such that  $\dim F_1 \geq \dim F_2 \geq \dots \geq \dim F_m$ .

Apply Theorem 5.2, we get the following.

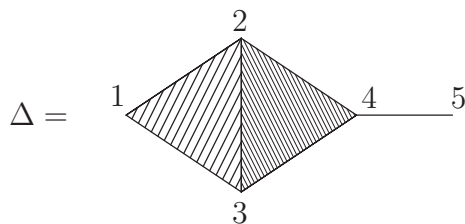
**Corollary 5.4.** *Under the same notation in Theorem 5.2. Suppose that  $|F_m| \geq 2$ . If  $\langle F_1, F_2, \dots, F_{i-1} \rangle \cap \langle F_i \rangle$  is a simplex for every  $2 \leq i \leq m$ , then  $\mathcal{R}(\mathbf{m})$  is a sequentially Cohen-Macaulay ring.*

Let us give some examples.

**Example 5.5.** Let  $\Delta = \langle F_1, F_2, F_3 \rangle$ , where  $F_1 = \{1, 2, 3\}$ ,  $F_2 = \{2, 3, 4\}$  and  $F_3 = \{4, 5\}$ . Then  $\Delta$  is shellable with shelling order  $F_1, F_2, F_3 \in \mathcal{F}(\Delta)$ . Then

$$\langle F_1 \rangle \cap \langle F_2 \rangle, \quad \langle F_1, F_2 \rangle \cap \langle F_3 \rangle$$

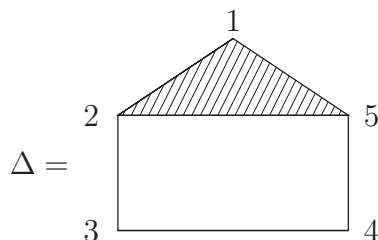
are simplexes, so that  $\mathcal{R}(\mathfrak{m})$  is a sequentially Cohen-Macaulay ring.



**Example 5.6.** Let  $\Delta = \langle F_1, F_2, F_3, F_4 \rangle$ , where  $F_1 = \{1, 2, 5\}$ ,  $F_2 = \{2, 3\}$ ,  $F_3 = \{3, 4\}$  and  $F_4 = \{4, 5\}$ . Notice that  $\Delta$  is a shellable simplicial complex with shelling order  $F_1, F_2, F_3, F_4 \in \mathcal{F}(\Delta)$ . We put  $\Delta_1 = \langle F_1, F_2, F_3 \rangle$  and  $\Delta_2 = \langle F_4 \rangle$ . Then

$$\sharp \mathcal{F}(\Delta_1 \cap \Delta_2) = 2 = |F_4|,$$

whence  $\mathcal{R}(\mathfrak{m})$  is not sequentially Cohen-Macaulay.



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